Exercise 1

Use residues to evaluate the improper integrals in Exercises 1 through 5.

$$\int_0^\infty \frac{dx}{x^2+1}.$$

Ans. $\pi/2$.

Solution

The integrand is an even function of x, so the interval of integration can be extended to $(-\infty, \infty)$ as long as the integral is divided by 2.

$$\int_0^\infty \frac{dx}{x^2+1} = \int_{-\infty}^\infty \frac{dx}{2(x^2+1)}$$

In order to evaluate the integral, consider the corresponding function in the complex plane,

$$f(z) = \frac{1}{2(z^2+1)},$$

and the contour in Fig. 93. Singularities occur where the denominator is equal to zero.

$$\begin{gathered} 2(z^2+1) = 0 \\ z^2+1 = 0 \\ z = \pm i \end{gathered}$$

The singular point of interest to us is the one that lies within the closed contour, $z = i$.

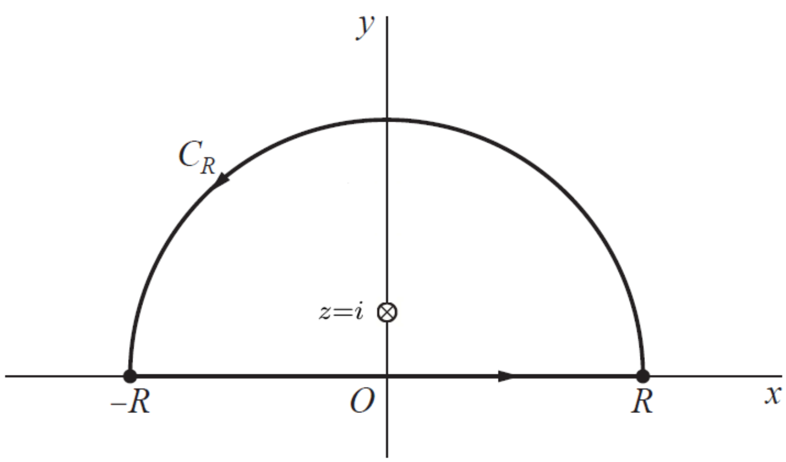

Figure 1: This is Fig. 93 with the singularity at $z = i$ marked.

According to Cauchy's residue theorem, the integral of $1/[2(z^2+1)]$ around the closed contour is equal to $2\pi i$ times the sum of the residues at the enclosed singularities.

$$\oint_C \frac{dz}{2(z^2+1)} = 2\pi i \operatorname*{Res}_{z=i} \frac{1}{2(z^2+1)}$$

This closed loop integral is the sum of two integrals, one over each arc in the loop.

$$\int_L \frac{dz}{2(z^2+1)} + \int_{C_R} \frac{dz}{2(z^2+1)} = 2\pi i \operatorname*{Res}_{z=i} \frac{1}{2(z^2+1)}$$

The parameterizations for the arcs are as follows.

$$\begin{aligned} L: &\quad z = r, && r = -R \quad \to \quad r = R \\ C_R: &\quad z = Re^{i\theta}, && \theta = 0 \quad \to \quad \theta = \pi \end{aligned}$$

As a result,

$$\int_{-R}^{R} \frac{dr}{2(r^2+1)} + \int_{C_R} \frac{dz}{2(z^2+1)} = 2\pi i \operatorname*{Res}_{z=i} \frac{1}{2(z^2+1)}.$$

Take the limit now as $R \to \infty$. The integral over C_R consequently tends to zero. Proof for this statement will be given at the end.

$$\int_{-\infty}^{\infty} \frac{dr}{2(r^2+1)} = 2\pi i \operatorname*{Res}_{z=i} \frac{1}{2(z^2+1)}$$

The denominator can be written as $2(z^2+1) = 2(z+i)(z-i)$. From this we see that the multiplicity of the $z-i$ factor is 1. The residue at $z = i$ can then be calculated by

$$\operatorname*{Res}_{z=i} \frac{1}{2(z^2+1)} = \phi(i),$$

where $\phi(z)$ is equal to $f(z)$ without the $z-i$ factor.

$$\phi(z) = \frac{1}{2(z+i)} \quad \Rightarrow \quad \phi(i) = \frac{1}{4i}$$

So then

$$\operatorname*{Res}_{z=i} \frac{1}{2(z^2+1)} = \frac{1}{4i}$$

and

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{dr}{2(r^2+1)} &= 2\pi i \left(\frac{1}{4i}\right) \\ &= \frac{\pi}{2}. \end{aligned}$$

Therefore, changing the dummy integration variable to x,

$$\boxed{\int_0^{\infty} \frac{dx}{x^2+1} = \frac{\pi}{2}.}$$

The Integral Over C_R

Our aim here is to show that the integral over C_R tends to zero in the limit as $R \to \infty$. The parameterization of the semicircular arc in Fig. 93 is $z = Re^{i\theta}$, where θ goes from 0 to π.

$$\begin{aligned}\int_{C_R} \frac{dz}{2(z^2+1)} &= \int_0^\pi \frac{Rie^{i\theta}\,d\theta}{2[(Re^{i\theta})^2+1]} \\ &= \int_0^\pi \frac{Rie^{i\theta}\,d\theta}{2(R^2e^{i2\theta}+1)}\end{aligned}$$

Now consider the integral's magnitude.

$$\begin{aligned}\left|\int_{C_R} \frac{dz}{2(z^2+1)}\right| &= \left|\int_0^\pi \frac{Rie^{i\theta}\,d\theta}{2(R^2e^{i2\theta}+1)}\right| \\ &\leq \int_0^\pi \left|\frac{Rie^{i\theta}}{2(R^2e^{i2\theta}+1)}\right| d\theta \\ &= \int_0^\pi \frac{\left|Rie^{i\theta}\right|}{|2(R^2e^{i2\theta}+1)|}\,d\theta \\ &= \int_0^\pi \frac{R}{|R^2e^{i2\theta}+1|}\frac{d\theta}{2} \\ &\leq \int_0^\pi \frac{R}{|R^2e^{i2\theta}|-|1|}\frac{d\theta}{2} \\ &= \int_0^\pi \frac{R}{R^2-1}\frac{d\theta}{2} \\ &= \frac{\pi}{2}\frac{R}{R^2-1}\end{aligned}$$

Now take the limit of both sides as $R \to \infty$.

$$\begin{aligned}\lim_{R\to\infty}\left|\int_{C_R} \frac{dz}{2(z^2+1)}\right| &\leq \lim_{R\to\infty}\frac{\pi}{2}\frac{R}{R^2-1} \\ &= \lim_{R\to\infty}\frac{\pi}{2R}\frac{1}{1-\frac{1}{R^2}}\end{aligned}$$

The limit on the right side is zero.

$$\lim_{R\to\infty}\left|\int_{C_R} \frac{dz}{2(z^2+1)}\right| \leq 0$$

The magnitude of a number cannot be negative.

$$\lim_{R\to\infty}\left|\int_{C_R} \frac{dz}{2(z^2+1)}\,dz\right| = 0$$

The only number that has a magnitude of zero is zero. Therefore,

$$\lim_{R\to\infty}\int_{C_R} \frac{dz}{2(z^2+1)}\,dz = 0.$$